

LINEAR CODES ON POSETS WITH EXTENSION PROPERTY

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ABSTRACT. We investigate linear and additive codes on partially ordered Hamming-like spaces that satisfy the extension property, meaning that automorphisms of ideals extend to automorphisms of the poset. The codes are naturally described in terms of translation association schemes that originate from the groups of linear isometries of the space. We address questions of duality and invariants of codes, establishing a connection between the dual association scheme and the scheme defined on the dual poset (they are isomorphic if and only if the poset is self-dual). We further discuss invariants that play the role of weight enumerators of codes in the poset case. In the case of regular rooted trees such invariants are linked to the classical problem of tree isomorphism. We also study the question of whether these invariants are preserved under standard operations on posets such as the ordinal sum and the like.

1. INTRODUCTION

The theory of linear codes is classically developed in the Hamming space over a finite field. Algebraic aspects of this theory are connected with the theory of association schemes, one of the main thrusts of which is related to duality theory of schemes and codes. While many results in this framework extend to additive codes over group alphabets, linear codes continue to be the main object of study. The focus of this work is poset metric spaces, i.e., finite spaces in which the distance is derived from partial orders coordinates. Poset metric spaces were introduced by Brualdi et al. [6] following the work of Niederreiter on one special case of this problem [22]. Extension of coding theory to poset metric spaces has been the subject of numerous publications in the last decade [4, 15, 17, 18, 26, 31].

Many basic theorems for linear codes are related to the notion of code duality which itself is derived from duality of the underlying association schemes. In this paper we are interested in duality of association schemes that arise from groups of linear isometries of poset metric spaces. Apart from being a convenient tool for the study of code duality [5, 9, 14, 21], group actions give rise to invariants of linear codes that are used in the study of structural and extremal properties of codes. These studies are particularly interesting when the underlying association scheme is self-dual. We show that group actions on poset metric spaces give rise to self-dual schemes if and only if the poset itself is self-dual. This proof relies on the structure of the isometry group of the poset metric space that was recently established by Panek et al. [26]. Another property that we involve is extension of automorphisms from order ideals to the entire poset; if this is possible, we say that the poset has the extension property.

Examples of posets studied in the literature are mostly confined to the hierarchical poset and the ordered Hamming space, defined below in the paper. Looking outside this set of examples, we consider posets whose Hasse diagrams are given by (level-)regular rooted trees. The poset weight is invariant under automorphisms, but this invariant is not sufficiently refined to characterize the orbits. Identifying such invariants leads us to the classical problem of encoding of rooted trees [28] and deciding

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isomorphism of trees [2]. Following [4, 21], we call such invariants shapes of codevectors. Finally, we consider standard operations to build new posets out of given ones, and we analyze the behavior of the extension property under those operations, making explicit the behavior of a shape when the extension property may be ensured. We remark that such operations enable us to construct self-dual posets from non-self-dual ones, extending the study of codes to new classes of posets.

We begin with some definitions and notation. Let \mathbb{F}_q^n be the n -dimensional linear space over \mathbb{F}_q . Let $\mathcal{P}(\llbracket n \rrbracket, \preceq)$ be a poset on $\llbracket n \rrbracket := \{1, 2, \dots, n\}$. A subset $I \subset \llbracket n \rrbracket$ is called an *ideal* of \mathcal{P} if the relations $i \in I, j \prec i$ imply that $j \in I$. An element $j \in I$ is called *maximal* if there are no elements $i \in I$ such that $j \prec i$. If i_1, i_2, \dots are the maximal elements of the ideal I , we say that I is generated by them, in the sense that I is the smallest ideal containing i_1, i_2, \dots and write $I = \langle i_1, i_2, \dots \rangle$. Denote by $M(I)$ the set of maximal elements of I . Define the *dual poset* \mathcal{P}^\perp on $\llbracket n \rrbracket$ by setting $i \prec j$ in \mathcal{P}^\perp whenever $j \prec i$ in \mathcal{P} . Ideals of \mathcal{P}^\perp are called *filters* of \mathcal{P} . We denote the set of all ideals and the set of all filters of \mathcal{P} by $\mathcal{I}(\mathcal{P})$ and $\mathcal{F}(\mathcal{P})$, respectively. Given a poset \mathcal{P} , on $\llbracket n \rrbracket$, we say that a subset $\{i_1, i_2, \dots, i_r\} \subseteq \llbracket n \rrbracket$ is a *chain of length r* if $i_1 \prec i_2 \prec \dots \prec i_r$. Given $i \in \llbracket n \rrbracket$, we define the *level* $l(i)$ of i as the maximal-length chain that has i as the maximal element.

Given a vector $x \in \mathbb{F}_q^n$ we define the support of x by $\text{supp}(x) = \{i : x_i \neq 0\}$. Let $\langle x \rangle$ denote the smallest-size ideal I such that $\text{supp}(x) \subseteq I$. Call $\omega_{\mathcal{P}}(x) = |\langle x \rangle|$ the *poset weight* of the vector $x \in \mathbb{F}_q^n$. Poset metric spaces were introduced in [6] where it is proved that $\omega_{\mathcal{P}}(x)$ is indeed a weight function, and so $d_{\mathcal{P}}(x, y) = \omega_{\mathcal{P}}(x - y)$ is a well-defined metric. The corresponding metric space will be denoted by $X = (\mathbb{F}_q^n, d_{\mathcal{P}})$. A *linear poset code* is a linear subspace of $(\mathbb{F}_q^n, d_{\mathcal{P}})$.

Poset metrics are invariant by translations, which makes them suitable for studying linear codes, since many of the aspects of the usual theory remain valid (the minimum distance being equal to the minimum weight, existence of syndrome decoding schemes, etc.). On the other hand, many metric results that hold for the Hamming space, can fail for a poset metric. For instance, the well known equation $\rho = \lfloor \frac{d-1}{2} \rfloor$, which relates the minimum distance d of a code with its packing radius ρ , not only is not valid for general posets, but those quantities may not at all to be related: there are linear codes with equal minimum distance but different packing radii [19].

Examples: 1. Let \mathcal{P} be an antichain on $\llbracket n \rrbracket$, i.e., no two elements of $\llbracket n \rrbracket$ are comparable. The metric $d_{\mathcal{P}}$ is the familiar Hamming distance of coding theory.

2. Let \mathcal{P} be a linear order (a chain), i.e., $1 \prec 2 \prec \dots \prec n$. The distance induced on $\llbracket n \rrbracket$ by \mathcal{P} has been studied in [13, 27, 31], while [25] gave a complete classification of linear codes in $(\mathbb{F}_q^n, \mathcal{P})$, establishing a canonical form for each class of such codes.

3. Suppose that $\llbracket n \rrbracket$ is a disjoint union $\llbracket n \rrbracket = H_0 \cup H_1 \cup \dots \cup H_m$ with the relation given by $i \prec j$ if and only if $i \in H_s, j \in H_t$ and $s < t$. This order defines a *hierarchical poset* \mathcal{P} on $\llbracket n \rrbracket$ which includes the above two examples as particular cases. The metric space $(\mathbb{F}_q^n, d_{\mathcal{P}})$ turns out to be the only instance of poset metrics for which the weight distribution of a linear code \mathcal{C} is determined by the weight distribution of its dual code \mathcal{C}^\perp [15, 17, 27].

4. Let $n = mr$ and let $\llbracket n \rrbracket$ be a disjoint union m of chains of length r . This example, which also includes the first two ones, is actually the first poset distance beyond the Hamming metric to be studied in combinatorics, see Niederreiter [22] and Rosenbloom and Tsfasman [29]. The arising metric space is called the ordered Hamming space or the NRT space. It finds applications in numerical analysis [23, 31] and coding theory [24]. Combinatorial structure of the NRT space was studied in detail in [1, 4, 21].

5. Let $n = 1 + d_0 + d_0 d_1 + \dots + d_0 d_1 \dots d_{m-2}$ and let $\llbracket n \rrbracket$ be a level-regular rooted tree in which every vertex i with $l(i) = s$ has d_s sons, $s = 0, 1, \dots, m-2$. Define \mathcal{P} by $i \prec j$ iff the vertex i lies on a path from the root to the vertex j . We will explore this example in detail in the next section (see Proposition 2.1), using a labelling of the vertices that is different from the encoding of trees considered in earlier literature beginning with [28]: there, each vertex is labelled individually with the purpose of arranging the vertices in a linear order, while we are concerned with the order \mathcal{P} .

Note also that encoding the nodes of a tree is related to the problem of genetic testing for hereditary diseases: namely, if an individual is found to be positive for a condition, this discovery supersedes the results of testing of his ancestors (which could have missed the presence of the same condition). If the family is represented by the ancestry tree, with each node labeled by 1/0 according as the individual tests positive or negative for this condition, then the metric on the family codewords is exactly the poset metric on a tree.

Poset metric automorphisms.

A poset automorphism is a permutation $\phi : \llbracket n \rrbracket \rightarrow \llbracket n \rrbracket$ such that $x \preceq y$ if and only if $\phi(x) \preceq \phi(y)$. Let $S_{\mathcal{P}}$ be the automorphism group of the poset $(\llbracket n \rrbracket, \mathcal{P})$. For instance, for the Hamming space $S_{\mathcal{P}} = \mathfrak{S}_n$ (the symmetric group on n elements) while for a single chain, $S_{\mathcal{P}} = \{\text{id}\}$. A poset \mathcal{P} is called *self-dual* if it is isomorphic to its dual \mathcal{P}^{\perp} , i.e., if there exists a permutation on $\llbracket n \rrbracket$ such that if $i \preceq_{\mathcal{P}} j$ then $\pi(j) \preceq_{\mathcal{P}^{\perp}} \pi(i)$. To give an example, let us represent \mathcal{P} by its *Hasse diagram* in which vertices correspond to elements of $\llbracket n \rrbracket$ and there is an edge connecting vertices i and j if and only if $i \prec j$ and $i \prec j' \preceq j$ implies $j' = j$.



Of the two posets in this figure the left is self-dual while the right is not. The NRT poset is self-dual, including the case of the single chain.

Since we are interested in linear codes, we concentrate on linear automorphisms of poset metric spaces. The group of linear isometries of $X = (\mathbb{F}_q^n, d_{\mathcal{P}})$, denoted $GL_{\mathcal{P}}(n)$, is formed of linear operators $T : X \rightarrow X$ such that $d_{\mathcal{P}}(T(x), T(y)) = d_{\mathcal{P}}(x, y)$ for all $x, y \in X$. The group $GL_{\mathcal{P}}(n)$ was characterized in [26]. It can be constructed as a semidirect product $G_{\mathcal{P}} \ltimes \text{Aut}(\mathcal{P})$ where $G_{\mathcal{P}}$ is the set of $n \times n$ matrices $A = (a_{ij})$ over \mathbb{F}_q such that

$$(1) \quad a_{ii} \neq 0, i = 1, \dots, n; \quad a_{ij} = 0 \text{ if } i > j; \quad a_{ij} = 0 \text{ if } i < j \text{ and } i \notin \langle j \rangle.$$

In particular, for the Hamming space, $GL_{\mathcal{P}}(n) = S_n \ltimes \mathbb{F}_q^*$, for the linear order we have $GL_{\mathcal{P}}(n) = M_n$, the group of upper-triangular matrices with nonzero main diagonal, and for the NRT space, $GL_{\mathcal{P}}(n) = M_r \ltimes S_m$ (the last result is due to [18]).

NOMENCLATURE:

- $\llbracket n \rrbracket = \{1, 2, \dots, n\}$
- \mathcal{P} – poset on $\llbracket n \rrbracket$
- $\omega_{\mathcal{P}}, d_{\mathcal{P}}$ – poset weight, poset distance
- $\mathcal{I}(\mathcal{P})$ the set of ideals of \mathcal{P} , $\mathcal{F}(\mathcal{P})$ the set of filters of \mathcal{P}
- $X = (\mathbb{F}_q, d_{\mathcal{P}})$ – poset metric space
- $\langle a_1, a_2, \dots \rangle$ – ideal generated by $a_1, a_2, \dots \in \llbracket n \rrbracket$
- $\text{supp}(x) = \{i : x_i \neq 0; x \in \mathbb{F}_q^n\}$
- $\langle x \rangle \triangleq \langle \text{supp}(x) \rangle$
- $M(I)$ – set of maximal elements of the ideal
- \mathcal{P}^{\perp} – dual poset of \mathcal{P}
- $GL_{\mathcal{P}}(n)$ – group of linear isometries of X
- $G_{\mathcal{P}}$ – subgroup of $GL_{\mathcal{P}}(n)$ that fixes the poset
- \tilde{I} – orbit of I under $\text{Aut}(\mathcal{P})$
- $l : \mathcal{P} \rightarrow \mathbb{Z}_+ \cup \{0\}$ – level (or rank) function for finite posets.

2. EXTENSION PROPERTY

Two ideals $I, J \in \mathcal{I}(\mathcal{P})$ are called isomorphic, denoted $I \sim J$, if there is a bijection $g : I \rightarrow J$ that preserves the order. Given a poset $\mathcal{P} = (\llbracket n \rrbracket, \preceq)$, an ideal $I \in \mathcal{I}(\mathcal{P})$, and a poset isomorphism

$\sigma \in \text{Aut}(\mathcal{P})$, it is clear that $\sigma(I) \sim I$ because $\sigma|_I$ is a poset isomorphism. However, the converse is not always true: given two isomorphic ideals $I, J \in \mathcal{I}(\mathcal{P})$ there does not always exist an automorphism $\sigma \in \text{Aut}(\mathcal{P})$ such that $\sigma(I) = J$. The simplest example of this situation is the poset on $\{1, 2, 3\}$ determined by the relation $1 \prec 3$; then $\{1\}$ and $\{2\}$ are isomorphic ideals, but $\text{Aut}(\mathcal{P}) = \text{Id}$.

Given a poset \mathcal{P} and an ideal $I \in \mathcal{I}(\mathcal{P})$, we denote by \tilde{I} the set of ideals that are isomorphic to I :

$$\tilde{I} = \{J \in \mathcal{I}(\mathcal{P}); I \sim J\}$$

We remark that \sim is an equivalence relation on $\mathcal{I}(\mathcal{P})$. In a similar way, given $I \in \mathcal{F}(\mathcal{P})$, we define

$$\tilde{I}^\perp = \{J \in \mathcal{F}(\mathcal{P}); I \sim J\}.$$

Definition 1. We say that a poset $\mathcal{P} = ([n], \preceq)$ has the ideal-extension (IE) property if, for every $I, J \in \mathcal{I}(\mathcal{P})$, if I and J are isomorphic, there exists $\sigma \in \text{Aut}(\mathcal{P})$ such that $\sigma(I) = J$. We say that $\mathcal{P} = ([n], \preceq)$ has the filter-extension (FE) property if the same holds true when ideals are replaced with filters.

A different and much stronger extension property (where the two sets $I, J \subseteq [n]$ need not to be ideals) was previously studied in the infinite case when such posets are called homogeneous [30], leading to classification of all such posets.

We say that $(\mathbb{F}_q^n, d_{\mathcal{P}})$ has the *orbits determined by ideals* (the \tilde{I} -property) if for any $x, y \in \mathbb{F}_q^n$ there is $T \in GL_{\mathcal{P}}(n)$ such that $T(x) = y$ if and only if $\langle x \rangle \sim \langle y \rangle$. When the orbits are determined by filters, we say that $(\mathbb{F}_q^n, d_{\mathcal{P}})$ has \tilde{I}^\perp -property.

In the next proposition we show that the \tilde{I} -property (a property of the vector space) and the IE-property (a property on the poset) are essentially the same:

Proposition 2.1. A poset \mathcal{P} has the IE-property iff $(\mathbb{F}_q^n, d_{\mathcal{P}})$ has the \tilde{I} -property. A poset \mathcal{P} has the FE-property iff $(\mathbb{F}_q^n, d_{\mathcal{P}})$ has the \tilde{I}^\perp -property.

Proof. Assume that \mathcal{P} has the IE-property. Let (e_i) be the standard basis of \mathbb{F}_q^n . As shown in [26, Theorem 1], given $T \in GL_{\mathcal{P}}(n)$, the map $\phi_T : [n] \rightarrow [n]$ defined by

$$\phi_T(i) = M(\langle T(e_i) \rangle),$$

is a poset automorphism. So, given $T \in GL_{\mathcal{P}}(n)$ such that $T(x) = y$, we have that ϕ_T is a poset automorphism and clearly $\phi_T(\langle x \rangle) = \langle y \rangle$, so that $\langle x \rangle \sim \langle y \rangle$. This establishes the only if part.

Suppose now that $\langle x \rangle \sim \langle y \rangle$. Since \mathcal{P} satisfies the extension property, there is $\phi \in \text{Aut}(\mathcal{P})$ such that $\phi(\langle x \rangle) = \langle y \rangle$. Let $T_\phi : \mathbb{F}_q^n \rightarrow \mathbb{F}_q^n$ be defined by $T_\phi(x_1, \dots, x_n) = (x_{\phi(1)}, \dots, x_{\phi(n)})$. Clearly, $T_\phi \in GL_{\mathcal{P}}(n)$.

By abuse of notation, we write $M(x)$ to refer to the set of maximal elements of $\langle x \rangle$. Given $x \in \mathbb{F}_q^n$, denote by $\hat{x} = (\hat{x}_1, \dots, \hat{x}_n)$ the vector that satisfies the following conditions: (i) $M(x) = M(\hat{x})$, (ii) $\text{supp}(\hat{x}) = M(x)$, (iii) If $i \in \text{supp}(\hat{x})$ then $\hat{x}_i = 1$.

Consider a matrix $A = (a_{ij}) \in G_{\mathcal{P}}$ such that $a_{ii} = x_i^{-1}$ if $i \in M(x)$, $a_{ii} = 1$ if $i \notin M(x)$ and $a_{ij} = 0$ for $i \neq j$. Let $Ax = (x'_1, \dots, x'_n)$, then $x'_i = 1$ if $i \in M(x)$ and $x'_i = x_i$ otherwise. Now consider the matrix $B = (b_{ij}) \in G_{\mathcal{P}}$ defined as follows:

$$\begin{aligned} b_{ij} &= 1 \text{ if } i = j \\ b_{ij} &= -x_i^{-1} \text{ if } x_i \neq 0 \text{ and } j = \max\{k \in M(x); i \prec k\} \\ b_{ij} &= 0 \text{ otherwise,} \end{aligned}$$

where \max refers to the usual order \leq of the natural numbers.

Let $T_x := BA$. By construction we have that $B \in G_{\mathcal{P}}$ and $BAx = \hat{x}$. Concluding, we have that $T = T_y^{-1} \circ T_\phi \circ T_x$ is a linear isometry that satisfies $T(x) = y$. ■

Posets in the five examples given above satisfy the IE property. For hierarchical posets, linear isometries act transitively on spheres of a fixed radius around zero. At the same time, for hierarchical posets, ideals are isomorphic if and only if they have the same cardinality, so that orbits of linear isometries are determined by the weight. Therefore, the IE-property is satisfied.

In contrast, the NRT posets also satisfy the IE property, but the cardinality of an ideal is not sufficient to characterize it, or equivalently, orbits of linear isometries are not determined by the weight. To do so, it is convenient to introduce a new invariant, *shapes of ideals* and observe that linear isometries act transitively on vectors of the same shape (more on this below).

We now show that the extension property also holds for level-regular rooted trees.

Proposition 2.2. *Level-regular rooted tree posets possess the IE property.*

Proof. Let $l(\cdot)$ be the rank function associated with the natural grading of \mathcal{P} . Let r be the height of the tree, i.e., $r = \max\{l(i) : i \in \llbracket n \rrbracket\}$ and suppose each element of rank $i < r$ has d_i descendants. Introduce a labeling of the vertices that associates a string of integers with a vertex $a \in \llbracket n \rrbracket$. Namely, if $l(a) = j \geq 1$, then the label $\lambda(a) = \alpha_1 \alpha_2 \cdots \alpha_j$, where $\alpha_m \in \{0, 1, \dots, d_m - 1\}$, $m = 1, \dots, j$. By definition $\lambda(\text{root}) = \emptyset$. The labeling is assigned in such a way that two elements $a, b \in \llbracket n \rrbracket$ with $\lambda(a) = \alpha_1 \alpha_2 \cdots \alpha_j$ and $\lambda(b) = \beta_1 \beta_2 \cdots \beta_k$ satisfy $a \preceq b$ iff $j \leq k$ and $\alpha_i = \beta_i$ for all $i = 1, 2, \dots, j$. In this case we can write $\lambda(b) = (\lambda(a) | \beta_{j+1} \cdots \beta_k)$. A labeling with such property is said to be *consistent with the order* \mathcal{P} . By abuse of notation, below we sometimes use labels to refer to vertices.

Let $I, J \in \mathcal{I}(\mathcal{P})$ be two isomorphic ideals, and let ϕ be the corresponding isomorphism. We are going to construct $\phi^* \in \text{Aut}(\mathcal{P})$ such that the restriction $\phi^*|_I = \phi$. For $a \in I$ set $\phi^*(a) = \phi(a)$. Given $a \in \llbracket n \rrbracket \setminus I$, consider the chain from the root to a . This chain is unique and intersects I because \mathcal{P} is a rooted tree. Let a_I be the last vertex in this chain that is in I (the “meet” of a and I). Thus we have $a_I = b_0 \prec b_1 \prec \cdots \prec b_{l(a)-l(a_I)} = a$ for some vertices $b_1, \dots, b_{l(a)-l(a_I)-1}$.

By our construction, the label of each b_l , $l \geq 1$ is obtained by concatenating the label $\lambda(a_I)$ with a tail formed of l letters β_1, \dots, β_l , where $\beta_j \in \{0, 1, \dots, d_{l(a_I)+j} - 1\}$, $j = 1, \dots, l$. For $a \in I$ define the set of descendants of a not contained in I :

$$\Lambda_{a,I} = \{0 \leq j \leq d_{l(a)} - 1 : (\lambda(a) | j) \notin I\}$$

(this set can be empty). Since I and J are isomorphic, and since \mathcal{P} is level-regular, we have

$$|\Lambda_{a,I}| = |\Lambda_{\phi(a),J}|, \quad a \in I,$$

so for each $a \in I$ there is a bijection $\gamma_a : \{0, 1, \dots, d_{l(a)} - 1\} \rightarrow \{0, 1, \dots, d_{l(\phi(a))} - 1\}$ such that $\gamma_a(\Lambda_{a,I}) = \Lambda_{\phi(a),J}$ and $(\lambda(\phi(a)) | \gamma_a(j)) = \lambda(\phi(\lambda(a) | j))$ for $j \in \{0, 1, \dots, d_{l(a)} - 1\} \setminus \Lambda_{a,I}$. In other words, γ_a induces the same map as ϕ when restricted to the immediate descendants of a in I .

Let us define the isomorphism ϕ^* as follows. Given $a \in \llbracket n \rrbracket$, consider its label

$$\lambda(a) = (\lambda(a_I) | \beta_{l(a_I)+1}, \dots, \beta_{l(a)})$$

and define $\phi^*(a)$ to be the vertex labeled as

$$\lambda(\phi^*(a)) = (\lambda(\phi(a_I)) | \gamma_a(\beta_{l(a_I)+1}), \beta_{l(a_I)+2}, \dots, \beta_{l(a)}).$$

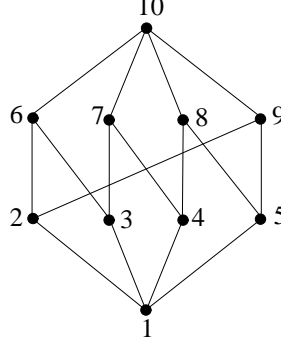
Since γ_a is a bijection, and since \mathcal{P} is level-regular, ϕ^* is well defined. It is a bijection that preserves the order because the labeling is consistent with \mathcal{P} . Therefore, it is an order isomorphism which also satisfies $\phi^*|_I = \phi$. ■

2.1. Remarks on lattices. A poset $\mathcal{P} = (\llbracket n \rrbracket, \prec)$ is called a *meet semilattice* if for any $x, y \in n$ there is a unique greatest lower bound z of x and y . We write $z = x \wedge y$ and call it the meet of x and y . Semilattices admit a natural grading, and we denote by X_i , $i = 0, 1, \dots, m$ its fibers, i.e., the sets of points of \mathcal{P} of the same rank. A semilattice $\mathcal{P} = (X, \prec)$ is said to be *regular* if the following conditions are satisfied:

- (1) Given $y \in X_m, z \in X_r$ with $z \preceq y$ the number of points $u \in X_s$ such that $z \preceq u \preceq y$ is a constant $\mu(r, s)$;

- (2) Given $u \in X_s$, the number of points $z \in X_r$ such that $z \preceq u$ is a constant $\nu(r, s)$;
- (3) Given $a \in X_r, y \in X_m$ with $a \wedge y \in X_j$, the number of pairs $(b, z) \in X_s \times X_m$ such that $b \preceq z, a \preceq z$ is a constant $\pi(j, r, s)$.

Regular semilattices were introduced by Delsarte [10]; see also [7], Ch.8. It is straightforward to show that level-regular rooted trees are regular semilattices and the family of semilattices seems to be a fertile ground for posets satisfying the IE property. The following example shows that not every regular semilattice satisfies the IE property.



Direct verification shows that \mathcal{P} is a regular meet semilattice. However, $I = \{1, 2, 3\}$ and $J = \{1, 2, 4\}$ are isomorphic ideals but there is no isomorphism of \mathcal{P} that maps I into J , since 2 and 3 are covered by 6, but no element in level 2 covers 2 and 4. This poset is also self-dual, and therefore forms a lattice.

For $a, b \in X$ define the set

$$a \vee b = \{x \in X : a \preceq x, b \preceq x \text{ and } (a \prec y \preceq x) \Rightarrow (y = x); (b \prec y \preceq x) \Rightarrow (y = x)\}.$$

We say that the semilattice \mathcal{P} is *strongly regular* if it is regular and satisfies the following additional conditions:

- (r1) For $z \in X_r, |\{u \in X_s; z \preceq u\}|$ is a constant $\bar{\nu}(r, s)$;
- (r2) Given a pair of vertices $a, b \in X_r$, if $a \vee b \neq \emptyset$, then the quantity $|\{u \in X_s; a \vee b \preceq u\}|$ is a constant $\rho(r, s)$ that does not depend on a and b .

It is not difficult to see that a level-regular rooted tree is a strongly regular semilattice, while the lattice in the previous example is not (indeed, let $r = 1$ and $s = 2$, then the pairs $(2, 3)$ and $(2, 4)$ give a counterexample to condition (r2)). We conjecture that strongly regular semilattices satisfy the IE property.

3. ASSOCIATION SCHEMES ON POSET METRIC SPACES

3.1. Association schemes on poset metric spaces. MacWilliams-type relations between weight enumerators of additive codes and their dual codes can be derived using Delsarte's theory of association schemes. We briefly summarize the facts about association schemes used below, following the presentation in [5, ch. 2]. Given a finite space X , a symmetric association scheme $\mathcal{A}(X, \mathcal{R} = (R_0, R_1, \dots, R_s))$ is a partition of the set $X \times X$ into s classes such that

- (i) $R_0 = \{(x, x), x \in X\}$,
- (ii) if $(x, y) \in R_\alpha$ then $(y, x) \in R_\alpha$ for all $x, y \in X, \alpha = 0, 1, \dots, s$.
- (iii) there are numbers $p_{\alpha\beta}^\gamma$ such that for any $(x, y) \in R_\gamma$ the number of $z \in X$ with $(x, z) \in R_\alpha$ and $(y, z) \in R_\beta$ equals $p_{\alpha\beta}^\gamma, \alpha, \beta, \gamma \in \{0, 1, \dots, s\}$.

The adjacency matrix A_α of the class R_α is defined by

$$(A_\alpha)_{xy} = \mathbf{1}_{(x,y) \in R_\alpha}, \quad \alpha = 0, 1, \dots, s$$

meaning that an entry $(x, y) \in A_\alpha$ is 1 if $(x, y) \in R_\alpha$ and 0 otherwise.

The matrices A_α generate an $(s + 1)$ -dimensional commutative \mathbb{C} -algebra called the adjacency algebra of \mathcal{A} . This algebra has a basis of primitive idempotents $(E_\alpha, \alpha = 0, 1, \dots, s)$. Define matrices P and Q , called the eigenvalues of \mathcal{A} , by

$$(2) \quad A_\beta = \sum_{\alpha=0}^s P_{\alpha\beta} E_\alpha \quad \text{and} \quad E_\beta = \frac{1}{|X|} \sum_{\alpha=0}^s Q_{\alpha\beta} A_\alpha.$$

The numbers $v_\alpha = p_{\alpha\alpha}^0$ and $m_\alpha = \text{rk } E_\alpha$ are called the valencies and the multiplicities of the scheme \mathcal{A} .

Suppose that X has the structure of an abelian group. A scheme \mathcal{A} is called a translation association scheme if for all $R \in \mathcal{R}$

$$(x, y) \in R \Rightarrow (x + z, y + z) \in R, \quad z \in X.$$

In this paper we restrict our attention to the case $X = \mathbb{F}_q^n$, an n -dimensional linear space.

The Hamming association scheme is defined by the relations $R_\alpha = \{(x, y) \in \mathbb{F}_q^n : d_H(x, y) = \alpha\}$. If \mathcal{P} is a single chain (Example 2) or an hierarchical poset (Example 3), we can again define $R_\alpha = \{(x, y) \in \mathbb{F}_q^n : d_{\mathcal{P}}(x, y) = \alpha\}$, $\alpha = 0, 1, \dots, n$ and prove that these relations satisfy (i)-(iii). However, for the NRT poset (Example 4) this approach fails to produce an association scheme. To define the NRT association scheme, also called the *ordered Hamming scheme*, let

$$(3) \quad \text{shape}(I) = (e_1, \dots, e_r), \text{ where } e_j = |\{i \in M(I) : l(i) = j\}|, j = 1, \dots, r.$$

Define $\mathcal{R} = \{R_e, e = (e_1, \dots, e_r)\}$, where e ranges over all the r -tuples of nonnegative integers such that $\sum_i e_i \leq m$. The relations of the scheme are given by $R_e = \{(x, y) \in (\mathbb{F}_q^n)^2 : \text{shape}(x - y) = e\}$. This scheme was introduced in [21] and further studied in [4]. We note that the group of linear isometries of the NRT space acts transitively on the sets $N_e = \{x \in \mathbb{F}_q^n : \text{shape}(x) = e\}$.

Let us consider a general poset metric space $X = (\mathbb{F}_q^n, d_{\mathcal{P}})$. There is more than one way to define an association schemes on X^1 . In the most general case, to every ideal I there corresponds a relation $R_I := \{(x, y) \in X^2 : \langle x - y \rangle = I\}$, although this definition is too general to be useful. We rely on the definition that is the most relevant to the theory of linear codes. Let $d_{\mathcal{P}}(\cdot, \cdot)$ be a poset metric on X and let $GL_{\mathcal{P}}(n)$ be the group of linear isometries. The action of $GL_{\mathcal{P}}(n)$ defines an equivalence relation on $X \times X$ where $x \sim y$ if the vectors x and y belong to the same orbit, i.e. there is $T \in GL_{\mathcal{P}}(n)$ such that $T(x) = y$. Let $\mathcal{X} := X/\sim$ be the set of orbits and suppose that $|\mathcal{X}| = s + 1$ for some s . Consider a partition $\mathcal{R} = \{R_\alpha | \alpha \in \mathcal{X}\}$ of $X \times X$ given by

$$(4) \quad R_\alpha = \{(x, y) \in X \times X | x - y \in \alpha\}, \quad \alpha \in \mathcal{X}.$$

The following simple observation sets the stage for the study of linear poset codes.

Proposition 3.1. *The pair (X, \mathcal{R}) forms a translation association scheme \mathcal{A} with s classes.*

Proof. We need to check the definition of the association scheme. The subsets $R_\alpha, \alpha \in \mathcal{X}$ form a partition because \sim is an equivalence relation on X and hence a partition of $X \times X$.

Property (i) follows because every $T \in G$ is invertible and linear, so $T(x) = 0$ iff $x = 0$.

Property (ii) (symmetry) follows because G is a group, and so $T \in G$ iff $T^{-1} \in G$.

Let us verify the intersection property (iii). Let (a_1, a_2) and (b_1, b_2) be representatives of the orbits α and β , respectively. Given $(x, y) \in R_\gamma$ denote

$$p_{\alpha\beta}^{(x,y)} = \{z \in X | (x, z) \in R_\alpha \text{ and } (y, z) \in R_\beta\}.$$

¹This is similar to classical coding theory: the action of the full group of linear isometries of the Hamming space defines the scheme relative to the Hamming weight, while the permutation part of the group gives rise to the scheme relative to complete weight enumerators, viz. [20], Sect.5.6.

Consider another pair $(x', y') \in R_\gamma$. There exists $T \in G$ such that $T(x - y) = x' - y'$. Let $z \in p_{\alpha\beta}^{(x,y)}$, then $(x, z) \in R_\alpha$ and $(y, z) \in R_\beta$. We shall prove that $(T(x), T(z)) \in R_\alpha$ and $(T(y), T(z)) \in R_\beta$. By definition of R_α there is $S \in G$ such that $S(x - z) = a_1 - a_2$. Then

$$T(x) - T(z) = T(x - z) = TS^{-1}(a_1 - a_2)$$

so $(T(x), T(z)) \in R_\alpha$. Similarly, $(T(y), T(z)) \in R_\beta$. It follows that there is an injective map $T : p_{\alpha\beta}^{(x,y)} \rightarrow p_{\alpha\beta}^{(x',y')}$. Similarly, T^{-1} defines an injective map in the reverse direction, and so $|p_{\alpha\beta}^{(x,y)}| = |p_{\alpha\beta}^{(x',y')}|$. ■

Another possible way to introduce an association scheme on (\mathbb{F}_q^n, d_P) is as follows. We consider on $\mathcal{I}(\mathcal{P})$ any equivalence relation and denote by \tilde{I} the isomorphism class of ideals that contains I . Let $\mathcal{R} = (R_{\tilde{I}})$, where $R_{\tilde{I}} = \{(x, y) \in (\mathbb{F}_q^n)^2 : \langle x - y \rangle \in \tilde{I}\}$. This definition was introduced in [14]; see also [8]. If the extension property holds true, this definition gives the same association scheme as the one defined in Proposition 2.1 above.

4. DUALITY OF SCHEMES

A translation association scheme \mathcal{A} has a dual scheme \mathcal{A}^* defined by the characters of the group X . Characters form a multiplicative group X^* with the operation given by $(\chi_1 \chi_2)(x) = \chi_1(x) \chi_2(x)$. It is well known that X and X^* are canonically isomorphic as groups. Let $\chi, \psi \in X^*$. Define the relations of the dual scheme by putting $(\chi, \psi) \in R_i^*$ iff $E_\alpha \eta = \eta$, where $\eta = \chi^{-1} \psi$. The dual translation scheme $\mathcal{A}^*(X^*, R^*)$ satisfies the following properties [5, p.69-70]:

(D1) Let v_α, m_α be the valencies and multiplicities of the scheme \mathcal{A} and let P, Q be its eigenvalues. The scheme \mathcal{A}^* is a translation association scheme with s classes, valencies $v_\alpha^* = m_\alpha$, multiplicities $m_\alpha^* = v_\alpha$, and eigenvalues $P^* = Q, Q^* = P$.

(D2) Let $N_\alpha = \{x \in X | (x, 0) \in R_\alpha\}$, $N_\alpha^* = \{\chi \in X^* | E_\alpha \chi = \chi\}$. Then $v_\alpha = |N_\alpha|$, $m_\alpha = |N_\alpha^*|$,

$$(5) \quad P_{\alpha\beta} = \sum_{x \in N_\beta} \chi(x), \quad \chi \in N_\alpha^*, \quad Q_{\alpha\beta} = \sum_{\chi \in N_\beta^*} \chi(x), \quad x \in N_\alpha.$$

(D3) $E_\alpha = \frac{1}{|X|} \sum_{\chi \in N_\alpha^*} \chi \chi^\dagger$, $\alpha = 0, 1, \dots, s$.

Two s -class association schemes $\mathcal{A}(X, \mathcal{R})$ and $\mathcal{B}(X', \mathcal{R}')$ are called *isomorphic* if there is a bijection $\phi : X \rightarrow X'$ such that $(x, y) \in R_\alpha$ if and only if $(\phi(x), \phi(y)) \in R'_{\pi(\alpha)}$ for some fixed permutation of the indices in \mathcal{R} and \mathcal{R}' . If \mathcal{A} and \mathcal{B} are translation schemes, then ϕ agrees with the translations.

4.1. Self-dual posets. Our motivation to study self-dual association schemes on posets comes from duality of linear codes. Let $\mathcal{C} \subset X$ be a linear code. The dual code of \mathcal{C} is the subgroup $\mathcal{C}^* = \{\chi \in X^* | \chi(x) = 1 \text{ for all } x \in \mathcal{C}\}$. Even though $X \cong X^*$, the codes \mathcal{C} and \mathcal{C}' live in different association schemes that are not necessarily isomorphic. In classical coding theory problems, the dual code \mathcal{C}^\perp is defined with respect to an inner product on $X \times X$, whereupon the codes \mathcal{C}^\perp and \mathcal{C}^* are identified with the help of the isomorphism of the dual groups. This is consistent for the Hamming scheme, but generally such identification does not necessarily preserve the scheme structure.

Example 2 (continued): Let $\mathcal{R} = \{R_0, R_1, \dots, R_n\}$, where $R_i = \{(x, y) : l(M(x - y)) = i\}$ for all i . The dual scheme can be realized by relations on characters that are defined by the dual chain $\mathcal{P}^\perp = (1 \succ 2 \succ \dots \succ n)$ [21].

Given a poset metric space X , define the scheme $\mathcal{A}^\perp = \mathcal{A}(X, \mathcal{R}_{\mathcal{P}^\perp})$ with respect to the action of the group $GL_{\mathcal{P}^\perp}(n)$. Duality of linear codes will be consistent with this definition if $\mathcal{A}^* \cong \mathcal{A}^\perp$. Natural candidates for this to hold are self-dual posets, in which case, of course, $\mathcal{A} \cong \mathcal{A}^\perp$.

This discussion motivates the following theorem.

Theorem 4.1. *Suppose that \mathcal{A} is a translation association scheme on X whose classes are given by orbits of the group $GL_{\mathcal{P}}(n)$ of linear isometries of a poset metric space (X, \mathcal{P}) . Then $\mathcal{A}^* \cong \mathcal{A}^\perp$ if and only if \mathcal{P} is self-dual.*

Proof: The “if” part follows straightforwardly from self-duality of \mathcal{P} . Formally, let $\alpha \in \mathcal{X}$ be an orbit of $GL_{\mathcal{P}}(n)$. For $x \in X$ denote by $\chi_x \in X^*$ its image under the isomorphism $X \cong X^*$. Let τ be a permutation on $\llbracket n \rrbracket$ that maps \mathcal{P} to \mathcal{P}^\perp and let T_τ be the corresponding $n \times n$ permutation matrix. The matrix $\phi = T_\tau$ defines a linear isometry on X .

We have $(\chi_x, \chi_y) \in R_\alpha^*$ iff $\eta := \chi_x^{-1}\chi_y$ is in eigenspace α , i.e., $E_\alpha\eta = \eta$, or, using (D3)

$$\frac{1}{|X|} \sum_{\chi \in N_\alpha^*} \chi \chi^\dagger \eta = \mathbf{1}(\eta \in N_\alpha^*),$$

i.e., iff $(x - y) \in \alpha$. Thus for the adjacency matrices of \mathcal{A}^* we have $(A_\alpha^*)_{xy} = \mathbf{1}((x - y) \in \alpha)$. At the same time, if \mathcal{A}^\perp is realized as the scheme on $\mathcal{I}(\mathcal{P}^\perp)$, then the orbit α is transformed into an isomorphic orbit $\phi(\alpha)$ with respect to the action of $GL_{\mathcal{P}}(n)$ on \mathcal{P}^\perp . Thus, we have $(A_{\phi(\alpha)}^\perp)_{xy} = \mathbf{1}((x - y) \in \alpha)$, i.e., $\mathcal{A}^* \cong_\phi \mathcal{A}^\perp$.

The “only if” part will follow from Proposition 4.3 below. Let \mathcal{Q} be a poset on $\llbracket n \rrbracket$ and let T be a linear isometry of the poset metric space $X = (\mathbb{F}_q^n, \mathcal{Q})$. Consider the poset metric space on $X_T = (\mathbb{F}_q^n, \mathcal{Q}_T)$ induced by \mathcal{Q} and T . Namely, given a vector $x = \sum_i x_i T(e_i)$ we define the weight $|x|_{\mathcal{Q}, T} := |\langle x \rangle_{\mathcal{Q}}|$.

Lemma 4.2. *The poset metric spaces X and X_T are isometric and thus the association schemes $\mathcal{A}_{\mathcal{Q}}$ and $\mathcal{A}_{\mathcal{Q}, T}$ are isomorphic.*

Proof: Indeed, given $x = \sum_i x_i e_i \in \mathbb{F}_q^n$ we have

$$\begin{aligned} \omega_{\mathcal{Q}, T}(T(x)) &= \omega_{\mathcal{Q}, T}\left(T\left(\sum_{j=1}^n \alpha_j e_j\right)\right) = \omega_{\mathcal{Q}, T}\left(\sum_{j=1}^n \alpha_j T(e_j)\right) = |\langle \{j | \alpha_j \neq 0\} \rangle_{\mathcal{Q}}| \\ &= \omega_{\mathcal{Q}}(x). \blacksquare \end{aligned}$$

Proposition 4.3. *Let \mathcal{P} and \mathcal{Q} be two posets defined on $\llbracket n \rrbracket$. Consider the metric spaces $X = (\mathbb{F}_q^n, d_{\mathcal{P}})$ and $X' = (\mathbb{F}_q^n, d_{\mathcal{Q}})$ and let $GL_{\mathcal{P}}(n)$ and $GL_{\mathcal{Q}}(n)$ be their groups of linear isometries. Suppose that the translation association schemes $\mathcal{A}_{\mathcal{P}}$ and $\mathcal{A}_{\mathcal{Q}}$ defined by these groups are isomorphic and the isomorphism $\phi : X \rightarrow X'$ is linear. Then the poset metric spaces $(\mathbb{F}_q^n, d_{\mathcal{P}})$ and $(\mathbb{F}_q^n, d_{\mathcal{Q}})$ are isometric and the posets \mathcal{P} and \mathcal{Q} are isomorphic.*

Proof. The proof is by induction on n . The base case $n = 1$ is straightforward. Suppose that the statement is true for every pair of posets \mathcal{P}' and \mathcal{Q}' defined on $\llbracket n - 1 \rrbracket$. Let \mathcal{P} and \mathcal{Q} be posets defined on $\llbracket n \rrbracket$ and suppose that $\mathcal{A}_{\mathcal{P}} \cong \mathcal{A}_{\mathcal{Q}}$. In other words, there is a linear bijection $\phi : X \rightarrow X$ such that $(x, y) \in R_{\mathcal{P}, \alpha}$ if and only if $(\phi(x), \phi(y)) \in R_{\mathcal{Q}, \pi(\alpha)}$ for some bijection π between the indices in $\mathcal{R}_{\mathcal{P}}$ and $\mathcal{R}_{\mathcal{Q}}$. Consider a basis $\beta = \{e_1, \dots, e_n\}$ of X such that $\text{supp}(e_i) = \{i\}$. Since ϕ is linear, it preserves linear independence, and it follows that $\{\phi(e_1), \dots, \phi(e_n)\}$ is linearly independent and hence a basis of X .

Denote by $\delta_{\mathcal{P}}(i, j)$ the graphical distance between i and j in the Hasse diagram of \mathcal{P} and let

$$J_{\mathcal{P}}^-(i) := \{j \in \llbracket n \rrbracket \mid \delta_{\mathcal{P}}(i, j) = 1, j \prec i\}.$$

Let i be a maximal element in \mathcal{P} and suppose w.l.o.g. that $i = n$. This can always be accomplished by isomorphically relabeling $\llbracket n \rrbracket$. Indeed, let us denote by θ such an isomorphism and let $T = T_\theta : \mathbb{F}_q^n \rightarrow \mathbb{F}_q^n$ be the linear map induced by θ : $T(\sum x_i e_i) = \sum x_i e_{\theta(i)}$. We note that the metric on the new poset \mathcal{P}' is induced by \mathcal{P} and T . Lemma 4.2 implies that (X, \mathcal{P}) and (X, \mathcal{P}_T) are isometric and give rise to isomorphic association schemes, so our assumption that n is maximal is justified. In a similar manner, we may assume that (again w.l.o.g.) that $j = n - 1 \in J_{\mathcal{P}}^-(n)$, or, in other words, that $n - 1 \in \langle n \rangle_{\mathcal{P}}$.

We claim that there is a \mathcal{P} -isometry that takes $e_n = (e_{n-1} + e_n) - e_{n-1}$ to $e_n - e_{n-1}$. Indeed, define an $n \times n$ matrix $M = (a_{ij})$ by follows:

$$\begin{aligned} a_{ii} &= 1, \quad \forall i \in \llbracket n \rrbracket \\ a_{n-1,n} &= -1 \\ a_{ij} &= 0 \quad \text{otherwise.} \end{aligned}$$

It is clear that $A(e_n) = e_n - e_{n-1}$. Moreover, since $n-1 \prec n$ we have that $A \in GL_{\mathcal{P}}(n)$ and is an isometry. Since $e_n = (e_{n-1} + e_n) - e_{n-1}$, we conclude that the pairs (e_{n-1}, e_n) and $(e_{n-1}, e_{n-1} + e_n)$ belong to the same relation $R_{\mathcal{P}, [e_n]} \in \mathcal{R}_{\mathcal{P}}$.

Letting $f_i = \phi(e_i)$ for all i , we conclude that the pairs (f_{n-1}, f_n) and $(f_{n-1}, f_n + f_{n-1})$ belong to the same class $R_{\mathcal{Q}, [f_n]} \in \mathcal{R}_{\mathcal{Q}}$. Therefore, there is an isometry that sends $f_n = (f_n + f_{n-1}) - f_{n-1}$ to $f_n - f_{n-1}$. Indeed, given a matrix $B = (b_{ij}) \in G_{\mathcal{Q}}$, we have that $B(f_n)$ has coordinates $(b_{1n}, b_{2n}, \dots, b_{n-1,n}, b_{nn})$ with respect to the basis $\{f_1, \dots, f_n\}$. Since the coordinates of $f_n - f_{n-1}$ are $(0, \dots, 0, 1, 1)$ and since $B \in G_{\mathcal{Q}}$ get that $n-1 \prec_{\mathcal{Q}} n$, or in other words, $n-1 \in J_{\mathcal{Q}}^-(n)$. The same reasoning can be applied to every $i \in J_{\mathcal{P}}^-(n)$, so that we find that $J_{\mathcal{P}}^-(n)$ is mapped into $J_{\mathcal{Q}}^-(n)$. Using the inverse isomorphism, we can conclude that actually this map between $J_{\mathcal{P}}^-(n)$ and $J_{\mathcal{Q}}^-(n)$ is bijective.

The induction hypothesis ensures that the posets $(\llbracket n-1 \rrbracket, \mathcal{P})$ and $(\llbracket n-1 \rrbracket, \mathcal{Q})$ are isomorphic. Since $J_{\mathcal{P}}^-(n) = J_{\mathcal{Q}}^-(n)$, we conclude that $(\llbracket n \rrbracket, \mathcal{P}) \cong (\llbracket n \rrbracket, \mathcal{Q})$. ■

4.2. Remarks on the parameters. Let $\mathcal{A} = (X, \mathcal{R})$ be a translation association scheme whose relations are indexed by the orbits of the isometry group of a poset metric space $X = (\mathbb{F}_q^n, c)$. The parameters of \mathcal{A} can be found from (5). Let $GL_{\mathcal{P}}(n)$ be the isometry group of X whose structure is given by (1). Let $N(x) = |Tx, T \in G_{\mathcal{P}}|$, and note that $N(x)$ depends only on $\langle x \rangle$. From (1) we obtain $N(x) = (q-1)^{|M(I)|} q^{|I \setminus M(I)|}$, where $I = \langle x \rangle$, and

$$v_{\alpha} = N(x) |\tilde{I}|.$$

The eigenvalues are found from (5) assuming that \mathcal{A}^* is realized on the metric space $X^{\perp} = (\mathbb{F}_q^n, d_{\mathcal{P}^{\perp}})$, i.e., that \mathcal{P} is self-dual. Namely, let $\alpha \in \mathcal{X}$ and let us fix any character $\chi \in N_{\alpha}^*$. Let $J := \langle \chi \rangle \in \mathcal{I}(\mathcal{P}^{\perp})$, by which we mean the ideal $\langle y \rangle_{\mathcal{P}^{\perp}}$, where $y \in \mathbb{F}_q^n$ is the image of χ under the isomorphism $X \cong X^*$. Consider the sum

$$P_{\alpha\beta} = \sum_{x \in N_{\beta}} \chi(x) = \sum_{x=(x_1, \dots, x_n) \in N_{\beta}} \prod_{i=1}^n \chi_{y_i}(x_i).$$

The terms with $x_i = 0$ or $y_i = 0$ contribute 1, so this sum is controlled by the intersection of the sets $I \cap J$ as subsets of $\llbracket n \rrbracket$. Moreover, if $y_i \neq 0$, then $\sum_{x_i \in \mathbb{F}_q^*} \chi_{y_i}(x_i) = 0$. Using the structure of the orbits (1), we obtain that the nontrivial contribution to the sum on x arises from the indices $i \in M(I) \cap J$. In this case $x_i \in \mathbb{F}_q^*$, and the sum on x_i is -1 . Summarizing, we obtain for the first eigenvalues of \mathcal{A} the expression

$$P_{\alpha\beta} = N(I) \sum_{I \in \tilde{\mathcal{I}}: (I \cap J) \subset M(I)} (-1)^{|M(I) \cap J|} (q-1)^{-|M(I) \cap J|}.$$

In the self-dual case, $Q = P$ and $m_{\alpha} = v_{\alpha}$ for all α . Generally, the first and second eigenvalues are connected by the well-known expression [5], Lemma 2.2.1(iv)

$$m_{\alpha} P_{\alpha\beta} = v_{\beta} Q_{\beta\alpha}$$

Let \mathcal{C} and \mathcal{C}^* be a pair of dual codes. The inner distributions are defined as vectors $a = (a_{\alpha}, \alpha \in \mathcal{X})$, $a_{\alpha} = |\mathcal{C} \cap N_{\alpha}|$ for all α , and $a' = (a'_{\alpha}, \alpha \in \mathcal{X}^*)$, $a'_{\alpha} = |\mathcal{C}^* \cap N_{\alpha}^*|$ for all α' . They are related by the MacWilliams equations [9]

$$(6) \quad a' = \frac{1}{|\mathcal{C}|} a Q, \quad a = \frac{|\mathcal{C}|}{q^n} a' P$$

Explicit computations of eigenvalues can be difficult since determining \tilde{I} is a hard task. Let us give one example.

Example 4 (continued): Consider again the NRT poset on $\llbracket n \rrbracket$, $n = mr$. The orbit of a vector x is formed of all vectors with a fixed shape (3), where $I = \langle x \rangle$. We have

$$v_e = \binom{m}{e_1, e_2, \dots, e_r} (q-1)^{\sum_{i=1}^r (i-1)e_i} q^{\sum_{i=1}^r e_i},$$

where $\binom{m}{e_1, e_2, \dots, e_r}$ is the number of ways of choosing e_i subsets of size $i = 1, \dots, r$ out of an m -set. It is known [5], Lemma 2.2.1(iv) that the eigenvalues satisfy orthogonality relations with weight v_e . This enables us to interpret the eigenvalues of the ordered Hamming scheme as r -variate orthogonal polynomials that belong to the family of multivariate Krawtchouk polynomials. This approach is further developed in [4].

Let us summarize our considerations of duality of codes and association schemes on poset metric spaces. First, there is no explicit need to realize \mathcal{A}^* on the dual poset. At the same time, if this can be done (in the case of self-duality), then both the code and its dual code can be visualized on the same Hasse diagram, which is convenient for their study. This explains why many previous studies that involved MacWilliams theorems for poset metrics [4, 11, 13, 21] dealt with self-dual posets. In such cases, the dual scheme \mathcal{A}^* is naturally identified with \mathcal{A}^\perp , which explains the switch of the ordering of the coordinates.

Next, if the poset is not self-dual, then duality of association schemes may have nothing to do with linear-algebraic duality of codes. In this case we still can derive MacWilliams-like relations between \mathcal{C} and \mathcal{C}^\perp , but they do not fit the original context of association schemes expressed by (6). This approach is taken in a recent work by Choi et al. [8] whose main purpose is to obtain such relations. Derivations in [8] still rely on characters, but those do not follow the structure of the dual scheme \mathcal{A}^* .

Note also the case of the hierarchical poset ([15–17], [27]), which generally is not self-dual. The association schemes \mathcal{A} and \mathcal{A}^\perp are not isomorphic, but the inner distributions nevertheless are compactly described in terms of the poset weight.

Our discussion can be summarized in the observation that in the non-self-dual case, MacWilliams-type relations generally come in two different, inequivalent forms related to the association schemes \mathcal{A}^* and \mathcal{A}^\perp .

5. SHAPES OF CODEVECTORS

MacWilliams relations are written with respect to the distribution of codevectors across the orbits of the group of linear isometries of the space (6). For instance, for the Hamming metric, two vectors have the same weight if and only if they are in the same orbit. In this context, the weight is a numerical invariant of vectors that characterizes the orbits. Generally, we call such a numerical invariant the *shape* of a vector.

Definition 2. Let $(\mathbb{F}_q^n, \mathcal{P})$ be a poset metric space. A mapping $s : \mathbb{F}_q^n \rightarrow \mathbb{Z}^n$ is called a *shape mapping* if it is constant on the orbits of $T \in GL_{\mathcal{P}}(n)$. The value that this mapping takes on the orbit of a vector $x \in \mathbb{F}_q^n$ is called the *shape* of x .

Shapes of vectors are known for only a small number of posets: hierarchical posets (see [12]), in which case they are given by the poset weight, and NRT-posets; see (3). The structure of the isometry group (1) suggests that shapes are determined by order ideals rather than vectors. This is the case for all the known examples. In particular, if \mathcal{P} has the IE-property, then the shape mapping depends only on the isomorphism class of ideals in the sense that $\text{shape}(x) = \text{shape}(y)$ if and only if $\langle x \rangle \sim \langle y \rangle$. Generally, the shape is difficult to determine, and we believe that there is no possibility of finding a general expression for it. Moreover, the shapes in the known examples not only determine the orbits,

but also other important invariants such as the weight of the vector and the packing radius of the 1-dimensional subspace generated by such vector. If shapes are rare, then such useful shapes, that can determine other invariants, are much more so.

Clearly, in the general case the shape is not uniquely defined. Moreover, generally it is difficult to check whether two vectors belong to the same orbit. We note that even in simple cases of the Hamming space and the single chain the order of the isometry group is exponential in n . In these cases, as well as in the case of a regular tree checking whether two vectors are isometric is easy. To be able to use shapes in the study of structural properties of codes in a poset space, we generally would like to be able to compute and compare shapes in time proportional to $\log |GL_{\mathcal{P}}(n)|$.

5.1. Codes on trees. In this section we consider metric spaces $X = (\mathbb{F}_q^n, \mathcal{P})$, where \mathcal{P} belongs to a class of posets whose Hasse diagrams are level-regular rooted trees, introduced in Example 5 above.

The group of linear isometries of X is given in (1). It is a semidirect product of $Aut(\mathcal{P})$ and the matrix group $G_{\mathcal{P}}$, where $Aut(\mathcal{P}) = S_{d_0} \times S_{d_0 d_1} \times \cdots \times S_{d_0 d_1 \dots d_{m-2}}$. Proposition 2.2 implies that shapes of vectors are determined by ideals, i.e., x and y are in the same orbit of $GL_{\mathcal{P}}(n)$ if and only if $\langle x \rangle \sim \langle y \rangle$. Thus, orbits are characterized by equivalence classes of ideals $\tilde{I}, I \in \mathcal{I}(\mathcal{P})$. Ideals of \mathcal{P} themselves are rooted trees (not necessarily regular), and their isomorphisms are obtained by restricting the isomorphisms of \mathcal{P} . We conclude that shapes of vectors in X will be determined if we find suitable numerical invariants of rooted trees that encode them up to isomorphism.

Isomorphism of rooted trees is a classical problem in computer science [28], [2]. There are many ways to encode a tree into a number so that it is possible to decode that number to an isomorphic tree (a representative of the same equivalence class). One way is as follows [28]. Let H be a tree of height h . Suppose all the vertices i with $l(i) = k + 1 \leq h$ are assigned labels, written as binary strings. Vertex j with $l(j) = k$ is given a label based on the labels of its sons. Suppose that j has d_k sons whose labels are C_1, C_2, \dots, C_{d_k} . These labels are sorted as binary numbers, so suppose that $C_1 \leq C_2 \leq \dots \leq C_{d_k}$, where identical labels are placed in arbitrary order. Then j is labeled as $(0|C_1|C_2|\dots|C_{d_k}|1)$. It remains to say what happens if the vertex has no sons: in this case it is assigned the label (01) . The label assigned to the root is the label of the tree. It is a binary word that can be also interpreted as an integer number. These labels play the role of shapes: two ideals have the same label if and only if they are isomorphic. The isometry group $GL_{\mathcal{P}}(n)$ acts transitively on all the binary vectors x whose support ideals $\langle x \rangle$ have the same label, and this label therefore can be used as the shape of the codevector.

The tree is not self-dual, so there is no well-defined duality of linear codes on it. At the same time, we can construct a self dual poset from a given poset \mathcal{P} by adjoining a mirror image to \mathcal{P} to \mathcal{P} . It is therefore of interest to study how the shapes behave under this and other similar operations on posets. We take up this question in the next section.

5.2. Operations on posets. There are several standard ways to create new posets from old. In addition to poset duality, the well-known operations include direct sums and direct products of posets as well as ordinal sums and products [32]. Suppose that we start with posets \mathcal{P} and \mathcal{Q} that have the IE property. We are interested whether this property is inherited by posets that arise as a result of combining \mathcal{P} and \mathcal{Q} , and in the positive case, what is the shape of codevectors on $\mathcal{P} * \mathcal{Q}$ given the shapes on \mathcal{P} and \mathcal{Q} (here $*$ is a generic notation for the operation).

Let $\mathcal{P} = (\llbracket n \rrbracket, \preceq_{\mathcal{P}})$ and $\mathcal{Q} = (\llbracket m \rrbracket, \preceq_{\mathcal{Q}})$ be two posets and let $\text{shape}_{\mathcal{P}}$ and $\text{shape}_{\mathcal{Q}}$ be their respective shape maps. Throughout this section \mathcal{P} and \mathcal{Q} are assumed to have the IE property.

5.2.1. Ordinal sum. Define a poset $\mathcal{P} \oplus \mathcal{Q}$ on $\llbracket m+n \rrbracket$ in the following way. Given $i, j \in \llbracket m+n \rrbracket$, put

$$i \preceq_{\oplus} j \iff \left\{ \begin{array}{l} i, j \leq n \text{ and } i \preceq_{\mathcal{P}} j \text{ or} \\ i, j > n \text{ and } (i-n) \preceq_{\mathcal{Q}} (j-n) \text{ or} \\ i \leq n \leq j \end{array} \right\}$$

We note that the hierarchical poset is an ordinal sum of several antichains. Also, ordinal sum gives us a way of making self-dual posets out of other posets. It turns out that of the operations considered in this

section, it is only ordinal sum that allows the IE property to be carried to the resulting poset from the component ones.

Lemma 5.1. *If both \mathcal{P} and \mathcal{Q} satisfy the IE-property, then so does $\mathcal{P} \oplus \mathcal{Q}$*

Proof. Given a subset $Y \subset \llbracket n + m \rrbracket$, consider the following sets:

$$\begin{aligned} Y_n &= \{i \in \llbracket m + n \rrbracket : i \in Y \text{ and } i \leq n\} \\ Y^m &= \{i \in \llbracket m + n \rrbracket : i > n\} \\ Y_m &= Y^m - n := \{i - n : i \in Y^m\}. \end{aligned}$$

Let us consider $I \in \mathcal{I}(\mathcal{P} \oplus \mathcal{Q})$. It is immediate to realize that $I^m \in \mathcal{I}(\mathcal{Q})$ and $I_n \in \mathcal{I}(\mathcal{P})$, and if $I_m \neq \emptyset$, then $I_n = \llbracket n \rrbracket$. Conversely, if $\emptyset \neq I \in \mathcal{I}(\mathcal{Q})$, then $\llbracket n \rrbracket \cup \{i + n \mid i \in I\}$ is an ideal in $\mathcal{P} \oplus \mathcal{Q}$, and if $I \in \mathcal{I}(\mathcal{P})$, then, viewed as a subset of $\llbracket n + m \rrbracket$, it is also an ideal.

Assume that $I, J \in \mathcal{I}(\mathcal{P} \oplus \mathcal{Q})$ and suppose there is a poset isomorphism $\phi : I \rightarrow J$. Naturally, we have that $\phi(I_n) = J_n$ and $\phi(I^m) = J^m$.

Suppose $I_m \neq \emptyset$ (and hence also $J_m \neq \emptyset$), then $I_n = J_n = \llbracket n \rrbracket$. Now note that I_m and J_m as subsets of $\llbracket m \rrbracket$ are ideals in \mathcal{Q} and the map $\tilde{\phi} : I_m \rightarrow J_m$ defined by $\tilde{\phi}(i) = \phi(i + n) - n$ is a poset isomorphism between I_m and J_m . By the IE-property of \mathcal{Q} , there is $\xi \in \text{Aut}(\mathcal{Q})$ such that $\xi(I_m) = J_m$. We define the map $\tilde{\phi} : \llbracket n + m \rrbracket \rightarrow \llbracket n + m \rrbracket$ as follows:

$$\begin{aligned} \tilde{\phi}(i) &= \xi(i - n) + n \text{ for } i > n \\ \tilde{\phi}(i) &= i \text{ for } i \leq n. \end{aligned}$$

Then we have that $\tilde{\phi} \in \text{Aut}(\mathcal{P} \oplus \mathcal{Q})$ and $\tilde{\phi}(I) = J$.

Suppose now that $I_m = J_m = \emptyset$. Then we may view I_n and J_n as isomorphic ideals in $\llbracket n \rrbracket \subset \llbracket n + m \rrbracket$. By the extension property of \mathcal{P} , there is $\xi \in \text{Aut}(\mathcal{P})$ such that $\xi(I_n) = J_n$. Define the map $\tilde{\phi} : \llbracket n + m \rrbracket \rightarrow \llbracket n + m \rrbracket$ as follows:

$$\begin{aligned} \tilde{\phi}(i) &= i \text{ for } i > n \\ \tilde{\phi}(i) &= \xi(i) \text{ for } i \leq n. \end{aligned}$$

Then we have that $\tilde{\phi} \in \text{Aut}(\mathcal{P} \oplus \mathcal{Q})$ and $\tilde{\phi}(I) = J$. ■

This lemma implies that, once shapes of vectors are defined both on \mathcal{P} and \mathcal{Q} , then in order to define a shape on $(\mathbb{F}_q^{n+m}, d_{\mathcal{P} \oplus \mathcal{Q}})$ it suffices to define it on $\mathcal{I}(\mathcal{P} \oplus \mathcal{Q})$. To simplify notation, we denote a shape on $\mathcal{P} \oplus \mathcal{Q}$ -shape by shape_{\oplus} .

Proposition 5.2. *Let $I \in \mathcal{I}(\mathcal{P} \oplus \mathcal{Q})$. Then the following mapping*

$$\text{shape}_{\oplus}(I) = \begin{cases} (0, \text{shape}_{\mathcal{Q}}(I_m)) & \text{if } I_m \neq \emptyset \\ (1, \text{shape}_{\mathcal{P}}(I_n)) & \text{if } I_m = \emptyset \end{cases}$$

is a shape on $\mathcal{P} \oplus \mathcal{Q}$.

Proof. Let $I, J \in \mathcal{I}(\mathcal{P} \oplus \mathcal{Q})$ be two ideals, and suppose that $\text{shape}_{\oplus}(I) = \text{shape}_{\oplus}(J)$. Suppose first that $I_m \neq \emptyset$, or equivalently, that $\text{shape}_{\oplus}(I) = (0, \text{shape}_{\mathcal{Q}}(I_m))$. Then we must have $\text{shape}_{\oplus}(J) = (0, \text{shape}_{\mathcal{Q}}(J_m))$ and thus $\text{shape}_{\mathcal{Q}}(I_m) = \text{shape}_{\mathcal{Q}}(J_m)$. It follows that I_m and J_m are isomorphic as posets, let us say by an isomorphism ϕ . Since $I_m \neq \emptyset$ we have that $I_n = J_n = \llbracket n \rrbracket$; hence $I = \llbracket n \rrbracket \cup (I)^m$ and $J = \llbracket n \rrbracket \cup (J)^m$. Define $\tilde{\phi} : I \rightarrow J$ by

$$\tilde{\phi}(i) = \begin{cases} i & \text{if } i \leq n \\ \phi(i - n) + n & \text{if } i > n \end{cases}$$

and notice that it is an isomorphism of posets, i.e., $I \sim J$. Now suppose that $I_m = \emptyset$. Then

$$(1, \text{shape}_{\mathcal{P}}(I_n)) = \text{shape}_{\mathcal{P}}(I) = \text{shape}_{\mathcal{P}}(J_n) = (1, \text{shape}_{\mathcal{P}}(J_n)).$$

From the IE-property of \mathcal{P} we obtain that $I_n \sim J_n$, and since $I = I_n, J = J_n$, we conclude that $I \sim J$.

Conversely, let us assume that $I \sim J$. Then we must have $I_n \sim J_n$ and $I^m \sim J^m$. Suppose that $I_m = \emptyset$. The IE-property ensures that $\text{shape}_{\mathcal{P}}(I_n) = \text{shape}_{\mathcal{P}}(J_n)$, and hence

$$\text{shape}_{\oplus}(I) = (1, \text{shape}_{\mathcal{P}}(I_n)) = (1, \text{shape}_{\mathcal{P}}(J_n)) = \text{shape}_{\oplus}(J)$$

Now suppose that $I_m \neq \emptyset$, then

$$I = \llbracket n \rrbracket \cup I^m, J = \llbracket n \rrbracket \cup J^m$$

and the isomorphism $\phi : I \rightarrow J$ maps I^m to J^m . It follows that the map $\tilde{\phi} : I_m \rightarrow J_m$ defined by $\tilde{\phi}(i) = \phi(i + n) - n$ is a poset isomorphism and by the IE-property we have that $\text{shape}_Q(I_m) = \text{shape}_Q(J_m)$, so that

$$\text{shape}_{\oplus}(I) = (0, \text{shape}_Q(I_m)) = (0, \text{shape}_Q(J_m)) = \text{shape}_{\oplus}(J)$$

■

5.2.2. Direct sum. Given \mathcal{P} and \mathcal{Q} , the direct sum operation results in a poset $\mathcal{P} + \mathcal{Q}$ in which the diagrams of \mathcal{P} and \mathcal{Q} are drawn “side by side.” Namely, given $i, j \in \llbracket n + m \rrbracket$ we have

$$i \preceq_{\oplus} j \iff \left\{ \begin{array}{l} i, j \leq n \text{ and } i \preceq_{\mathcal{P}} j \text{ or} \\ i, j > n \text{ and } (i - n) \preceq_{\mathcal{Q}} (j - n) \end{array} \right\}$$

This poset does not inherit the IE property from \mathcal{P} and \mathcal{Q} . To give a simple example, suppose that \mathcal{P} and \mathcal{Q} are not isomorphic. Then we pick a \mathcal{P} -minimal element $i \in \llbracket n \rrbracket$ and a \mathcal{Q} -minimal element $j \in \llbracket m \rrbracket$. The sets $\{i\}$ and $\{j + n\}$ are trivially isomorphic as ideals in $\mathcal{P} \oplus \mathcal{Q}$, but there is no poset automorphism on $\mathcal{P} \oplus \mathcal{Q}$ that would exchange them.

5.2.3. Ordinal product. Given posets $\mathcal{P} = (\llbracket n \rrbracket, \preceq_{\mathcal{P}})$ and $\mathcal{Q} = (\llbracket m \rrbracket, \preceq_{\mathcal{Q}})$, the poset $\mathcal{P} \otimes \mathcal{Q} = (\llbracket n \rrbracket \times \llbracket m \rrbracket, \preceq_{\otimes})$ is defined by the relations

$$(i, j) \preceq_{\otimes} (i', j') \iff i = i' \text{ and } j \preceq_{\mathcal{Q}} j'.$$

The IE property is not inherited from \mathcal{P} and \mathcal{Q} . Indeed, suppose \mathcal{P} is a poset on $\{1, 2\}$ defined by the relations $i \preceq_{\mathcal{P}} i, i = 1, 2$. Suppose that \mathcal{Q} is a hierarchical poset having at least two minimal elements, let us say i and j , and assume that they are not maximal, i.e., there is $k \in \llbracket m \rrbracket$ such that $i \preceq_{\mathcal{Q}} k$ and $j \preceq_{\mathcal{Q}} k$. Then the sets $\{(1, i), (1, j)\}$ and $\{(1, i), (2, i)\}$ are ideals (since $1, 2, i, j$ are all minimal elements), and since neither i and j nor 1 and 2 are comparable, those ideals are isomorphic. At the same time, there is no $\phi \in \text{Aut}(\mathcal{P} \otimes \mathcal{Q})$ that takes $\{(1, i), (1, j)\}$ to $\{(1, i), (2, i)\}$. Indeed, $(1, i) \preceq_{\otimes} (1, k)$ and $(1, j) \preceq_{\otimes} (1, k)$, but there is no (x, y) such that $(1, i) \preceq_{\otimes} (x, y)$ and $(2, i) \preceq_{\otimes} (x, y)$ since $(1, i) \preceq_{\otimes} (x, y)$ implies that $x = 1$.

5.2.4. Direct product. Given posets $\mathcal{P} = (\llbracket n \rrbracket, \preceq_{\mathcal{P}})$ and $\mathcal{Q} = (\llbracket m \rrbracket, \preceq_{\mathcal{Q}})$, the poset $\mathcal{P} \times \mathcal{Q} = (\llbracket n \rrbracket \times \llbracket m \rrbracket, \preceq_{\times})$ is defined by the relation

$$(i, j) \preceq_{\times} (i', j') \iff i \preceq_{\mathcal{P}} i' \text{ and } j \preceq_{\mathcal{Q}} j'.$$

It is easy to see that a subset $I \times J \subset \llbracket n \rrbracket \times \llbracket m \rrbracket$ is an ideal of $\mathcal{P} \times \mathcal{Q}$ if and only if $I \in \mathcal{I}(\mathcal{P})$ and $J \in \mathcal{I}(\mathcal{Q})$. Moreover, $\phi = (\phi_{\mathcal{P}}, \phi_{\mathcal{Q}}) \in \text{Aut}(\mathcal{P} \times \mathcal{Q})$ if $\phi_{\mathcal{P}} \in \text{Aut}(\mathcal{P})$ and $\phi_{\mathcal{Q}} \in \text{Aut}(\mathcal{Q})$. At the same time, clearly not any $\phi \in \text{Aut}(\mathcal{P} \times \mathcal{Q})$ can be expressed in such a way. It follows that the extension property does not necessarily hold on $\mathcal{P} \times \mathcal{Q}$, as can be seen in the following example.

Example: Let $\mathcal{P} = (\llbracket 2 \rrbracket, \preceq_{\mathcal{P}})$ and $\mathcal{Q} = (\llbracket 3 \rrbracket, \preceq_{\mathcal{Q}})$ be defined by the relations $1 \prec_{\mathcal{P}} 2, 1 \prec_{\mathcal{Q}} 2$, and $1 \prec_{\mathcal{Q}} 3$. Then $\mathcal{P} \times \mathcal{Q}$ is generated by the relations

$$\begin{aligned} (1, 1) \preceq_{\times} (2, 1); (1, 1) \preceq_{\times} (1, 2); (1, 1) \preceq_{\times} (1, 3) \\ (2, 1) \preceq_{\times} (2, 2); (1, 2) \preceq_{\times} (2, 2); (1, 2) \preceq_{\times} (2, 3); (1, 3) \preceq_{\times} (2, 3). \end{aligned}$$

The sets

$$I = \{(1, 1), (1, 2)\}, J = \{(1, 1), (2, 1)\}$$

are both ideals in $\mathcal{P} \times \mathcal{Q}$ that are isomorphic as posets, but there is no $\phi \in \text{Aut}(\mathcal{P} \times \mathcal{Q})$ such that $\phi(I) = J$.

Some of the initial ideas of this paper appeared earlier in the extended abstract [3]. Here they are both developed and refined in a number of ways.

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